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Note on the Settling of Small Particles in a Recirculating Flow

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IN the formation of platelet aggregates or microemboli in flows containing a vortex-like separated region, the residence time of a platelet (or other small particle) within the separated part of the flow is of prime importance. One factor which could limit the residence time is the tendency of small particles to settle under the influence of gravity. In connection with this problem, we have evaluated a more general situation: the kinematics of the settling of a small particle in any recirculating flow characterized by two-dimensional closed streamlines in a vertical plane. The particle is assumed to have constant settling velocity corresponding to a small sphere moving under the influence of gravity at terminal velocity, and to have $v_s \ll u$, where v_s is the magnitude of the settling velocity and u that of the flow. For this situation, one might expect that the vertical distance displaced during a complete circulation to be simply the product of the settling velocity and the circulation time. The calculation presented here shows that this is not so, and that to a good approximation, the particle returns to its original streamline in a complete circulation, and the net vertical distance displaced is zero.

The particle motion is taken to be that of the background flow with the added constant settling velocity. The particle starts at some point on a streamline ψ_1 and moves to adjacent streamlines due to the settling motion in the negative y direction. (The x-axis is horizontal.)

Let ds be a small length along ψ_1. In traversing ds, the particle will settle a distance dy below ψ_1 given by

$$dy = -v_s(ds/u), \quad dy \ll ds \tag{1}$$

corresponding to a change in stream function

$$d\psi = (\partial\psi/\partial y)dy = u_x dy = -u_x(v_s ds/u) = -v_s(u_x/u\, ds) \tag{2}$$

or

$$d\psi = -v_s dx$$

Thus the net change in ψ during the flow with a constant v_s, is simply

$$\psi_2 - \psi_1 = -v_s(x_2 - x_1) \tag{3}$$

The result, Eq. (3), is valid to first order in v_s/u and is independent of the details of the flow. It states that when $x_2 = x_1$ then $\psi_2 = \psi_1$, and hence a particle returns to its original streamline twice in a complete circulation, and does *not* settle out of the flow. A particle "falls" inward during one part of the flow and outward during the other, the two excursions completely cancelling; independent of the details of the flow. A typical trajectory is shown in Fig. 1. (The points P and Q in the figure have the same x coordinate and indicate positions where the particle trajectory crosses the streamline ψ_1.)

The result, Eq. (3), also implies that, to first order in (v_s/u), the displacement δ normal to an initial streamline is:

$$\delta = (v_s/u)\Delta x \tag{4}$$

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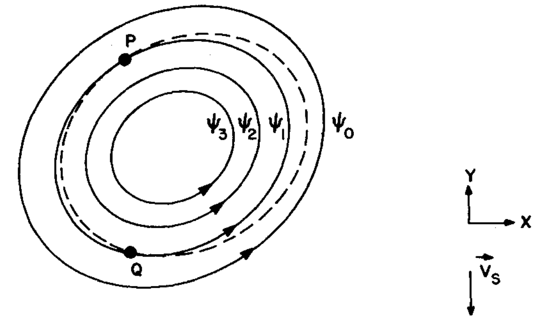

Fig. 1 Trajectory (dotted line) of a particle as it "settles" in a recirculating flow.

since $\Delta\psi = u\delta$. Thus for situations involving a vortex-like region surrounded by boundaries, (such as walls, or dividing streamlines in a separated flow) settling can affect only those particles which lie on streamlines which pass within δ of a boundary as given by Eq. (4), with Δx determined by the projection of the streamline path on the x axis. Since we are taking $v_s \ll u$, the streamlines affected represent only a small part of the flowfield. Thus, to first order in (v_s/u), settling due to gravity should be unimportant for particles in any two-dimensional recirculating flow in a vertical plane.

Localized Diamond-Shaped Buckling Patterns of Axially Compressed Cylindrical Shells

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Introduction

THE upper stability limit of cylindrical shells with classical simply supported boundary conditions was treated long ago by Flügge[1] and Pflüger.[2] This upper limit, as is well known, fails to serve as a design criterion due to the snap-through effect (Durchschlag) common in shell structures. An explanation of this snap-through characteristic and an approximate solution of this problem has been given by von Kármán et al.[3] This work which is now a classic, has been developed by many authors[2,4,5] and has also been further refined and improved in recent times, notably by Esslinger.[4] Because of the mathematical difficulty and the large algebraic labor involved, the majority of these works have dealt with simply supported boundary conditions and an over-all buckling pattern to keep the amount of calculation in reasonable limits. Indeed, the only exception known to the author is that of Ref. 5 where, unfortunately, no results were obtained because of the convergence difficulties in the large computer program.

In the following we have, therefore, the rather humble aim of obtaining only an estimation of the lower stability limit of the free edge cylinder. The method is based on a simple differential equation with Pogorelov coefficients.[6] Mechanically, the method can be interpreted as the buckling of a strut on a "bending" foundation with an attenuated eigenvector. With "bending," it is meant that the elastic constant of this fictitious foundation is due to the isometric bending in the circumferential direction.[9]

Energy Functional and the Isometric Transformation

In general, a shell surface is a two-dimensional Riemann space for which the integrability conditions are given by the

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Mainardi-Codazzi and Gauss equations

$$b_{\alpha\beta}|_{\gamma} = b_{\alpha\gamma}|_{\beta} \tag{1}$$

$$R_{\varepsilon\alpha\beta\gamma} = b_{\alpha\beta}b_{\gamma\varepsilon} - b_{\alpha\gamma}b_{\beta\varepsilon} \tag{2}$$

where $b_{\alpha\beta}$ and $R_{\varepsilon\alpha\beta\gamma}$ are the second fundamental tensor and the Riemann curvature tensor respectively. For instance, in the immediate vicinity of any point on a shallow sphere, it can easily be shown that the first equation is identically, and the second equation is approximately, satisfied. Consequently, a small part of an arbitrary shell surface can always be regarded as quasi-developable. On the other hand, the deformation of a quasi-developable surface is essentially isometric. Thus, in accordance with the Gauss principle of the least constraint,[10] the local deformation will be dominated by the internal isometric energy[6] at least in the advanced post buckling range. Recalling the known experimental fact that shell buckling is a local phenomenon, the preceding discussion forms a simple and logical interpretation of shell buckling in general and eliminates the discrepancy between the boundary conditions and the isometric deformation of the shell surface.

Consider the simple energy functional of a cylindrical shell under axial compression as given for instance in Ref. 7. Thus, we have

$$V = \int_0^l [C_1\ddot{\omega}^2 + C_2\omega^2 + P\dot{\omega}^2]2\pi r\,dx \tag{3}$$

where $C_1 = [Et^3/24(1-\nu^2)]$, $C_2 = Et/2r^2$, t is the shell thickness, r is the shell radius, l is the length of the shell, E is the modulus of elasticity, ν is the Poisson ratio, P is the axial pressure, ω is the radial displacement, and $(\dot{\ }) = d(\ )/dx$. On the grounds of the preceding discussion this functional, which corresponds to an ordinary linear differential equation, can be used to determine the lower stability limit of the post buckling path. This method thus offers considerable advantages and simplifications when compared with the conventional one where the solution of a non-linear partial differential equation is normally required.[3] Now, the simplest way of performing an isometric transformation of the potential energy functional ($V \to I$) is changing C_i in Eq. (3) as follows

$$C_1 \to k_1 = [E(t\pi)^3/12(1-\nu^2)b^3]; \quad C_2 \to k_2 = (Etb^5/1440r^2) \tag{4}$$

where b is the circumferential wavelength of the isometric transformation.[6,9] Thus, we have

$$I = \int_0^l \left[\frac{Etb^5}{1440r^2(1-\nu^2)}\ddot{\omega}^2 + \frac{E(t\pi)^3}{12(1-\nu^2)b^3}\omega^2 + \frac{P}{2}\dot{\omega}^2\right]2\pi r\,dx \tag{5}$$

In other words, we are treating the advanced asymmetrical (final) configuration as a pseudo bifurcation of a strut on a "bending" elastic foundation. That is, the stiffness of this fictitious foundation is due to the inextensional bending of the circumferential elements rather than the membrane hoop stresses. Using the calculus of variations, a corresponding differential equation with the so called Pogorelov coefficients[6]† is easily generated from the previous energy functional.

Boundary Conditions

We consider the case of a free-edge cylindrical shell. Further, we assume that the cylinder is semi-infinite which can easily be shown to be justified a posteriori. Thus, the bending moment M and the shear force Q are

$$Q = M = 0 \to x = 0$$

and for $x \to \infty$ we require that ω, $\dot{\omega}$ is bounded.

Upper Bound and Exact Solution

Considering the preceding boundary conditions, a very simple solution can be obtained when assuming the following attenuated buckling form for ω

† The derivation of these coefficients is not given here as it can easily be obtained in an elementary way identical to that of Kirste.[9] For more details, the reader is referred to a comprehensive treatise by Pogorelov.[6]

$$\omega = a\,e^{-ix}\cos ix \tag{6}$$

where a is the radial edge deflection, and i is a damping constant. Since Eq. (6) satisfies the relevant geometrical boundary conditions, a simple Rayleigh-Ritz procedure using Eq. (6) will give an upper bound solution. Inserting Eq. (6) in Eq. (5) and evaluating some improper integrals, we get

$$P^c = (i^2Etb^5/1080) + [E(t\pi)^3/12i^2(1-\nu^2)b^3] \tag{7}$$

Minimizing P^c with respect to i we obtain the lower critical stress (Durchschlag stress)

$$\sigma^c_{\min} \cong 0.1069\frac{Et}{r(1-\nu^2)} \tag{8}$$

It is now of interest to examine the influence of satisfying both the geometrical and the dynamical boundary conditions. Taking the following buckling mode

$$\omega = 2a\,e^{-\gamma x}[(3)^{1/2}\cos\delta x - \sin\delta x] \tag{9}$$

where

$$\gamma = K/(2)^{1/2}, \quad \delta = K(\tfrac{3}{2})^{1/2}, \quad K = \left[\frac{30t^2\pi^3r^2}{b^8(1-\nu^2)}\right]^{1/4}$$

which is, incidentally, the exact eigenvector of the corresponding differential equation and should therefore, give the exact value, we obtain

$$\sigma^c_{\min} = 0.093\frac{Et}{r(1-\nu^2)} \tag{10}$$

This differs from Eq. (8) by only 15%.

We could probably demonstrate that the preceding method and the underlaying mechanical model is a reasonable approximation by looking at the simply supported cylinder. The results of this case differ very slightly from that of Volmir[8]

$$\left(\sigma^c_{\min} = 0.186\frac{Et}{r(1-\nu^2)}\right)$$

and are in full agreement with Kirste's ingenious solution.[9] It might be interesting to compare Eq. (10) with some experimental values which have been evaluated statistically and quoted in Ref. 8 (p. 544). For $r/t = 1000$ and a probability of 90 to 99%, the experimental buckling load was found to be

$$\sigma^c = 0.13\frac{Et}{r(1-\nu^2)} \sim \frac{0.08Et}{r(1-\nu^2)} \tag{11}$$

This is in excellent agreement with our analytically obtained result, Eq. (10).

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